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Entropy in (1 + 1)-Dimensional Black Hole

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The Klein–Gordon equation and Dirac equation are solved in the backgrounds of a (1 + 1)-dimensional black hole with 'tHooft and "quasiperiodic" boundary conditions, respectively. The corresponding entropies of bosons and fermions are calculated; the divergence in the fermionic entropy has the same form as that in the bosonic one, except that the coefficient is different.

1. INTRODUCTION

Entropy can be described both in a thermodynamic sense and in terms of a counting of states. In the traditional thermodynamic sense, the area of the event horizon of a black hole is interpreted as its thermodynamic entropy, the surface gravity on the horizon is proportional to the Hawking temperature, and the classical Bekenstein–Hawking entropy is proportional to the area of horizon and satisfies all thermodynamic laws (Bekenstein, 1972, 1973, 1974; Hawking, 1975; Kallosh *et al.*, 1993). The matter field fluctuations originating from the black hole background is an interesting problem ('tHooft, 1985; Susskind and Uglum, 1994). 'tHooft calculated the number of scalar particle states surrounding a black hole in a so-called "brick wall model" and found the quantum scalar field fluctuation about the Hartle–Hawking temperature. By ignoring the contribution from the system surrounded by vacuum, he gave the following one-loop contribution to the entropy:

$$S_{sch}^{q} = \frac{8\pi^3}{45} \frac{(2M)^4}{h\beta^2}$$

where β is the reciprocal of the Hawking temperature, and h is a cutoff.

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Here S^q_{sch} is regarded as a geometric quantity (Solodukin, 1995a) $A_h/48\pi\varepsilon^2$, with ε the ultraviolet cutoff, and $\varepsilon^2 = \frac{15}{2}\delta^2$; $\delta = 2\sqrt{r_h h}$ is the proper distance from the horizon r_h to $r_h + h$. A different but actually equivalent approach (Callan and Wilczek, 1994; Kabat and Strassler, 1994) is adopted by Bombelli *et al.* (1986) and Srednicki (1993). In 'tHooft's brick wall model, as well as in Sloldukhin's use of the Gibbons–Hawking Euclidean path integral approach (Gibbons and Hawking, 1977) to study the quantum corrections to the entropy of a Schwarzschild black hole starting with the one-loop effective action of scalar matter, there exists a logarithmic divergence $\sim\log(\Lambda/\varepsilon)$, where Λ is the infrared cutoff. The divergence in entropy arises because of an infinite number of states which appear on the horizon. In quantum mechanics, the geometric entropy satisfies the following assumptions: If particles are scalar bosons obeying Bose–Einstein statistics, the entropy obtained is conventionally called the bosonic entropy. If the quantum mechanical geometric entropy is calculated by counting the fermionic particle states, the corresponding entropy is called the fermionic entropy.

Recently, such problems have attracted much interest ('tHooft, 1985; Solodukin, 1995a,b; Ghosh and Mitra, 1994, 1995; Russo, 1995; Fila *et al.*, 1994; de Alwis and Ohta, 1995; Zhou *et al.*, 1995; Hawking, 1995; Ichinose and Satoh, 1995; Larsen and Wilczek, 1996; Shen *et al.*, 1997); different approximations have been used to study the quantum corrections to the entropy in various black hole backgrounds.

In this paper, we shall solve directly the Klein–Gordon equation and Dirac equation in $(1 + 1)$ dimensional black hole backgrounds with 'tHooft and "quasiperiodic" boundary conditions and calculate the corresponding bosonic entropy and fermionic entropy, respectively. The results show that the fermionic entropy has completely the same divergence form as that of bosonic entropy, except that the coefficient is different.

2. THE BOSONIC ENTROPY OF A (1 + 1)-DIMENSIONAL BLACK HOLE

The following metric is taken for the black hole (Achuarro and Ortiz, 1993):

$$dS^2 = N^2 dt^2 - \frac{1}{N^2}\, dr^2 \tag{1}$$

where

$$N^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \qquad -\infty < t < +\infty, \quad 0 < r < +\infty$$

M and J are both integral constants, l is a constant concerned with the cosmological constant: $l^{-2} = -\Lambda$, and Λ is the negative cosmological constant. The horizons of the black hole are given by

$$r_{\pm}^2 = \frac{1}{2} Ml^2 \left\{ 1 \pm \left[1 - \left(\frac{J}{Ml} \right)^2 \right]^{1/2} \right\} \tag{2}$$

Here it is defined that $J < Ml$. The surface gravity of the black hole is

$$k = \frac{r_+^2 - r_-^2}{r_+ l^2} \tag{3}$$

Considering the massless scalar field wave equation in the background spacetime (1), which is

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \psi) = 0 \tag{4}$$

and substituting (1) into (4), one has

$$\frac{1}{N^2} \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial}{\partial r} \left(N^2 \frac{\partial \psi}{\partial r} \right) = 0 \tag{5}$$

Let

$$\psi(t, r) = U(r) \exp(-iEt) \tag{6}$$

Using (6), one can obtain from (5)

$$\frac{1}{N^2} E^2 U + \frac{\partial}{\partial r} \left(N^2 \frac{\partial U}{\partial r} \right) = 0 \tag{7}$$

Let

$$dy = \frac{dr}{N^2} \tag{8}$$

One then has

$$-\frac{\partial^2 U}{\partial y^2} = E^2 U \tag{9}$$

Hence we have

$$U(y) = \exp(\pm iEy) \tag{10}$$

From (8) we have

$$y = 4l^2 \ln\left[\left(\frac{4r - |g|}{4r + |g|}\right)^{-\frac{\sqrt{|g|}}{4h}}\left(\frac{4r - |f|}{4r + |f|}\right)^{\frac{\sqrt{|f|}}{4h}}\right] \tag{11}$$

Here

$$h = 4Ml^2\sqrt{1 - \left(\frac{J}{Ml}\right)^2}, \qquad f = -2Ml^2\left[1 + \sqrt{1 - \left(\frac{J}{Ml}\right)^2}\right],$$
$$g = -2Ml^2\left[1 - \sqrt{1 - \left(\frac{J}{Ml}\right)^2}\right]$$

Using the 'tHooft boundary condition

$$U(r_+ + \delta) = U(r_+ + L) \tag{12}$$

one thus has the eigensolution

$$U(r) = \sin\left\{\frac{l^2}{h} E \ln\left[\left(\frac{\frac{4r - |g|}{4r + |g|}}{\frac{4r_+ + 4\delta - |g|}{4r_+ + 4\delta + |g|}}\right)^{-\sqrt{|g|}} \cdot \left(\frac{\frac{4r - |f|}{4r + |f|}}{\frac{4r_+ + 4\delta - |f|}{4r_+ + 4\delta + |f|}}\right)^{\sqrt{|f|}}\right]\right\} \tag{13}$$

Since

$$\begin{aligned}
4r_+ + 4\delta - |g| &= 4Ml^2\left[1 - \left(\frac{J}{Ml}\right)^2\right]^{1/2} + 4\delta \\
4r_+ + 4\delta + |g| &= 4Ml^2 + 4\delta \\
4r_+ + 4\delta - |f| &= 4\delta \\
4r_+ + 4\delta + |f| &= 4Ml^2\left\{1 + \left[1 - \left(\frac{J}{Ml}\right)^2\right]^{1/2}\right\} + 4\delta
\end{aligned} \tag{14}$$

(13) can be written as

$$U(r) = \sin\left\{\frac{l^2}{h} E \ln\left[\left(\frac{\dfrac{4r - |g|}{4r + |g|}}{\dfrac{Ml^2\left[1 - \left(\dfrac{J}{Ml}\right)^2\right]^{1/2} + \delta}{Ml^2 + \delta}}\right)^{-\sqrt{|g|}} \cdot \left(\frac{\dfrac{4r - |f|}{4r + |f|}}{\dfrac{\delta}{Ml^2\left\{1 + \left[1 - \left(\dfrac{J}{Ml}\right)^2\right]^{1/2}\right\} + \delta}}\right)^{\sqrt{|f|}}\right]\right\} \tag{15}$$

while the eigenvalue is

$$\frac{l^2}{h} E \ln W = n(E)\pi \tag{16}$$

with W defined to be

$$W = \left[\left(\frac{\dfrac{Ml^2\left[1 - \left(\dfrac{J}{Ml}\right)^2\right]^{1/2} + L}{Ml^2 + L}}{\dfrac{Ml^2\left[1 - \left(\dfrac{J}{Ml}\right)^2\right]^{1/2} + \delta}{Ml^2 + \delta}}\right)^{-\sqrt{|g|}} \cdot \left(\frac{\dfrac{L}{Ml^2\left\{1 + \left[1 - \left(\dfrac{J}{Ml}\right)^2\right]^{1/2}\right\} + L}}{\dfrac{\delta}{Ml^2\left\{1 + \left[1 - \left(\dfrac{J}{Ml}\right)^2\right]^{1/2}\right\} + \delta}}\right)^{\sqrt{|f|}}\right]$$

According to the standard statistical approach, the free energy of the scalar field is

$$\beta F = \sum_n \ln[1 - e^{-\beta E(n)}]$$
$$= -\beta \int dE \frac{n(E)}{e^{\beta E} - 1} \tag{17}$$

Here $\beta = 1/k_B T$, T is the Hawking temperature, and k_B is the Boltzmann constant.

By direct calculation one obtains

$$F = -\frac{\pi}{6\beta^2} \frac{l^2}{h} \ln W \tag{18}$$

The entropy corresponding to (18) is

$$S_b = \beta^2 \frac{\partial F}{\partial \beta}$$
$$= \frac{l^2}{h} \ln W \tag{19}$$

Substituting $\beta = 1/k_B T$, $T = \kappa/2\pi k_B$, $\kappa = (r_+^2 - r_-^2)/r_+ l^2$ into (19), one gets

$$S_b = \frac{1}{24 r_+} \ln W \tag{20}$$

Here S_b is just the bosonic entropy of the black hole, the metric of which is given in (1), δ is the ultraviolet cutoff, and L is the infrared cutoff.

3. THE FERMIONIC ENTROPY OF A (1 + 1)-DIMENSIONAL BLACK HOLE

In this section, we calculate the fermionic entropy of the black hole, the metric of which is given in (1).

Let us consider massless spinor particles with two complex components in (1 + 1)-dimensional space-time. The action is (Solodukhin, 1995c; Mielke *et al.*, 1993)

$$I_f = \int \frac{i}{2} \varepsilon_{ab} e^a \wedge (\bar{\psi} \gamma^b \nabla \psi - \nabla \bar{\psi} \gamma^b \psi) \tag{21}$$

Here $e^a = e^a_\mu dx^\mu$, $a = 0, 1$. The 2-dimensional metric of the curved surface $\mathcal{M}^2$ can be written as $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, while $\eta_{ab} = (+1, -1)$; the Lorentz connection 1-form is $\omega^a_b = \omega \varepsilon^a_b$, $\omega = \omega_\mu dx^\mu$, while $\varepsilon_{ab} = \varepsilon_{ba}$, $\varepsilon_{01} = 1$. The matrix γ^a satisfies the relationships

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_5 = \gamma^0\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{22}$$

and

$$\gamma^a\gamma_5 + \gamma_5\gamma^a = 0$$

In 2-dimensional complex spinor space, a covariant spinor derivative ∇, which is a differential operator acting on the ψ field, can be defined as

$$\nabla\psi = d\psi + \tfrac{1}{2}\omega\gamma_5\psi, \qquad \nabla\overline{\psi} = d\overline{\psi} - \tfrac{1}{2}\omega\overline{\psi}\gamma_5 \tag{23}$$

Using the calculus of variations for (21), one obtains the fermion field equation as

$$\varepsilon_{ab}[e^a \wedge \gamma^b d\psi - d(e^a\gamma^b\psi)] = 0 \tag{24}$$

Let

$$\psi(t, r) = \exp(-iEt)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{25}$$

Using (1) and (25), we find that (24) becomes

$$N^2 \frac{d\psi_1}{dr} + \frac{1}{4}\frac{dN^2}{dr}\psi_1 + iE\psi_1 = 0 \tag{26}$$

$$N^2 \frac{d\psi_2}{dr} + \frac{1}{4}\frac{dN^2}{dr}\psi_2 - iE\psi_2 = 0 \tag{27}$$

From (26) and (27), we get the following eigensolutions:

$$\psi_1 = C_1\left[-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right]^{-1/4} \exp\left\{-iE\frac{l^2}{h}\ln\left[\left(\frac{4r - |g|}{4r + |g|}\right)^{-\sqrt{|g|}} \cdot \left(\frac{4r - |f|}{4r + |f|}\right)^{\sqrt{|f|}}\right]\right\} \tag{28}$$

$$\psi_2 = C_2\left[-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right]^{-1/4} \exp\left\{iE\frac{l^2}{h}\ln\left[\left(\frac{4r - |g|}{4r + |g|}\right)^{-\sqrt{|g|}} \cdot \left(\frac{4r - |f|}{4r + |f|}\right)^{\sqrt{|f|}}\right]\right\} \tag{29}$$

Here C_1, C_2 are integral constants.

In order to calculate the fermionic entropy of the black hole, the so-called "quasiperiodic" boundary condition is introduced (Zhou *et al.*, 1995),

$$\sqrt{N}\psi_j\big|_{r_+ +\delta} = \sqrt{N}\psi_j\big|_{r_+ +L}, \qquad j = 1, 2 \tag{30}$$

On the other hand, it is required that the phase factor of the fermionic eigensolution must satisfy the periodicity condition, hence we get

$$\frac{l^2}{h} E \ln W = 2n\pi \tag{31}$$

According to Fermi–Dirac statistics, the one-module partition function is

$$\begin{aligned} Z(n) &= \sum_{m=0}^{2} e^{-m\beta E(n)} \\ &= (1 - e^{-3\beta E})(1 - e^{-\beta E})^{-1} \end{aligned} \tag{32}$$

The free energy then is

$$F = -\beta^{-1} \sum_n \ln[(1 - e^{-3\beta E(n)})(1 - e^{-\beta E(n)})^{-1}] \tag{33}$$

Since

$$\beta F = \int \ln\left[\frac{1 - e^{-\beta E(n)}}{1 - e^{-3\beta E(n)}}\right] dn(E)$$

the expression for the entropy is

$$S = \beta^2 \frac{\partial F}{\partial \beta} \tag{34}$$

and similarly to the bosonic case, we obtain

$$S_f = \frac{1}{72r_+} \ln W \tag{35}$$

S_f is just the fermionic entropy of the black hole whose metric is given in (1).

Comparing (36) with (20), it is obvious that the fermionic entropy and the bosonic entropy have almost the same form, except that the coefficients are different.

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